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# Representations of the exceptional and other Lie algebras with integral eigenvalues of the Casimir operator

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**Abstract**
The uniformity, for the family of exceptional Lie algebras $\mathfrak{g}$, of the decompositions of the powers of their adjoint representations is now well known for powers up to four. The paper describes an extension of this uniformity for the totally antisymmetrized $n$th powers up to $n = 9$, identifying (see tables 3 and 6) families of representations with integer eigenvalues $5, \ldots, 9$ for the quadratic Casimir operator, in each case providing a formula (see equations (11)–(15)) for the dimensions of the representations in the family as a function of $D = \dim \mathfrak{g}$. This generalizes previous results for powers $j$ and Casimir eigenvalues $j$, $j \leqslant 4$. Many intriguing, perhaps puzzling, features of the dimension formulae are discussed and the possibility that they may be valid for a wider class of not necessarily simple Lie algebras is considered.

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## 1. Introduction

After noting some conventions in section 1.1, we describe quite carefully the context of this paper in section 1.2. This enables us to indicate briefly in section 1.3 the scope of this paper, and highlight the new results it obtains.

### 1.1. Notation and conventions

We are concerned with simple complex Lie algebras $\mathfrak{g}$ and with their irreducible representations. Irreducibility is understood over the field of complex numbers. We note that we use the informal abbreviation irrep for irreducible representation.

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We focus, in particular, on the family of

$$\mathcal{F} = \{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{g}_2, \mathfrak{d}_4, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\} \tag{1}$$

of simple Lie algebras. They feature in the extension of the last line of the Freudenthal magic square [1] that is given in [2]. These algebras are well known to form a family in some profound sense whose ramifications probably have not yet been fully exhausted.

Our work depends heavily on access to a large body of data for the Lie algebras $\mathfrak{g}$, especially lists for the exceptional Lie algebras of irreducible representations $R$ classified by highest weights which give the corresponding dimension and eigenvalue $c^{(2)}(R)$ of the quadratic Casimir operator $\mathcal{C}^{(2)}(R)$. We have created a C++ program to provide this and related information given the Cartan matrix as the only input. We also note that valuable general sources of data regarding Lie algebras are available, e.g., [3, 4].

We use a normalization in which $c^{(2)}(\mathfrak{g}) = 1$ for the adjoint representation and therefore $c^{(2)}(R) = \langle \Lambda_R, \Lambda_R + 2\delta \rangle$ where $\Lambda_R$ denotes the highest weight of $R$, $\delta$ is the half-sum of positive roots of $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle$ denotes the Cartan–Killing form on the space of weights.

We refer to irreps here often by their dimension because our studies are concerned with dimension formulae for families of representations of Lie algebras $\mathfrak{g}$. When we need to refer to irreps by their highest weight or Dynkin coordinate specification, we adopt the conventions that follow from the Cartan matrices of $\mathfrak{g}$ used by [3–5]. We also often omit commas between the coordinates, here always integers less than ten.

The diagram automorphisms of the algebras $\mathfrak{g} \in \mathcal{F}$ are $Z_2$ for $\mathfrak{a}_2$ and $\mathfrak{e}_6$, $S_3$ for $\mathfrak{d}_4$, and the trivial group for all the others. As the adjoint irrep $ad$ is always mapped to itself under diagram automorphisms, the constituents of the complete decomposition of its tensor powers $ad^{\otimes j}$ are either self-conjugate or pairs of complex conjugate irreps for $\mathfrak{a}_2$ and $\mathfrak{e}_6$. For $\mathfrak{d}_4$, the constituents are either irreps that are stable under triality or triples and sextuples of irreps that are related by triality.

### *1.2. Background material*

The first property of the family $\mathcal{F}$ to be noted concerns the structure of $ad \otimes ad$. We write this as

$$ad \otimes ad = (ad \otimes ad)_A \oplus (ad \otimes ad)_S. \tag{2}$$

For the antisymmetric piece we have a universal result, i.e. one that is valid for each simple compact $\mathfrak{g}$,

$$(ad \otimes ad)_A = ad \oplus X_2 \tag{3}$$

where $X_2$ denotes a representation of $\mathfrak{g}$ of dimension

$$\dim X_2 = \tfrac{1}{2} D(D-3) \tag{4}$$

where $D = \dim \mathfrak{g}$. For $\mathfrak{a}_2$, $X_2$ is the representation $20 = 10 + \overline{10} = (3, 0) \oplus (0, 3)$, a pair of conjugate irreps. For the exceptional Lie algebras, see table 1.

A universal and an important property of the family of irreps $X_2$ is the result

$$c^{(2)}(X_2) = 2. \tag{5}$$

It is of special relevance to the work described here, because families $X_j$ with

$$c^{(2)}(X_j) = j \tag{6}$$

for $j \leqslant 4$ are known to appear in the $j$th antisymmetric tensor power of $ad$, and we extend this knowledge beyond $j = 4$ here.

**Table 1.** Irreps of $\mathfrak{g}$ for $ad \otimes ad$.

| | $\mathfrak{g}_2$ | $\mathfrak{f}_4$ | $\mathfrak{e}_6$ | $\mathfrak{e}_7$ | $\mathfrak{e}_8$ |
|---|---|---|---|---|---|
| $ad$ | 14 | 52 | 78 | 133 | 248 |
| | (10) | (1000) | (000001) | (1000000) | (00000010) |
| $X_2$ | 77′ | 1274 | 2925 | 8645 | 30380 |
| | (03) | (0100) | (001000) | (0100000) | (00000100) |
| $R_1$ | 1 | 1 | 1 | 1 | 1 |
| $R_2$ | 27 | 324 | 650 | 1539 | 3875 |
| | (02) | (0002) | (100010) | (0000100) | (10000000) |
| $R_3$ | 77 | 1053 | 2430 | 7371 | 27000 |
| | (20) | (2000) | (000002) | (2000000) | (00000020) |

**Table 2.** The complete reduction of the representations $X_3$ and $X_4$ of $\mathfrak{g} \in \mathcal{F}$ in Dynkin coordinates and the results $d_3(\dim \mathfrak{g})$ and $d_4(\dim \mathfrak{g})$ of the dimension formulae (9) and (10).

| | $\mathfrak{a}_1$ | $\mathfrak{a}_2$ | $\mathfrak{g}_2$ | $\mathfrak{d}_4$ | $\mathfrak{f}_4$ | $\mathfrak{e}_6$ | $\mathfrak{e}_7$ | $\mathfrak{e}_8$ |
|---|---|---|---|---|---|---|---|---|
| $d_3$ | −5 | 0 | 182 | $3 \cdot 840$ | 19448 | 70070 | 365750 | 2450240 |
| $X_3$ | (4) | – | (04) | (0022) ⊕(2002) ⊕(2020) | (0020) | (010100) | (0010000) | (00001000) |
| $d_4$ | 0 | −35 − 35 | 0 | $3 \cdot 3675$ | 205751 | $2 \cdot 600600$ | 11316305 | 146325270 |
| $X_4$ | – | (14) ⊕ (41) | – | (1013) ⊕(1031) ⊕(3011) | (0021) | (020010) ⊕(100200) | (0001001) | (00010000) |

The result corresponding to (3) for the symmetric piece of $ad \otimes ad$ is not universal, but for each $\mathfrak{g} \in \mathcal{F}$ we have a result of the form

$$(ad \otimes ad)_S = R_1 \oplus R_2 \oplus R_3 \tag{7}$$

defining three families of irreps as given in table 1.

These families enjoy a variety of nice properties: for each family we have single formulae for the dimension and for the $\mathcal{C}^{(2)}$ eigenvalue of its members as a function of $D = \dim \mathfrak{g}$. We do not need these details here. As far as we can determine, the first full explicit analysis of $ad \otimes ad$ appears in [6].

The analysis just discussed for $ad \otimes ad$ gives rise to a natural conjecture—the Deligne conjecture [7]—that the $j$th tensor powers of $ad$ for the exceptionals possess uniform decompositions into irreps. That this does indeed happen, defining further families of irreps, has been comprehensively confirmed by algebraic computation in [8], using [9], for $j = 3, 4$ and established independently of computational procedures in [10].

We do not review this in full, but note, for $j = 3$, only the result

$$ad^{\wedge 3} = (ad \otimes ad \otimes ad)_A = ad \oplus R_2 \oplus R_3 \oplus X_2 \oplus X_3 \tag{8}$$

valid for all $\mathfrak{g} \in \mathcal{F}$. The important fact here is that (8) defines only one new family beyond those already understood from the study of $ad \otimes ad$ which we denote by $X_3$ (table 2). The irreps involved here possess two notable properties, natural analogues of (4) and (5). Their dimension is given by a polynomial in $D$,

$$d_3(D) = \frac{1}{3!} D(D-1)(D-8) \tag{9}$$

and equation (6) holds. The factor $D - 8$ in (9) reflects the fact that $\mathfrak{a}_2$ has no irrep with $c^{(2)} = 3$ and hence no member in the $X_3$ family. For $\mathfrak{a}_1$, the structure of (8) also collapses, and there is no representation with $c^{(2)} = 3$ in $ad^{\wedge 3}$ either. However, for $D = 3$, the dimension formula (9) gives the answer $-5$, and $\mathfrak{a}_1$ does indeed have a representation of dimension 5 with $c^{(2)} = 3$ which is, however, not contained in $ad^{\wedge 3}$. No systematic understanding has been obtained of why negative values of the dimension formula are often seen to happen and seem to make some sort of sense. More examples follow in our work for $j = 5, \ldots, 9$. Finally, we remark that the three irreps that occur for $\mathfrak{d}_4$ (table 2) are related by diagram automorphisms.

Next we note that analysis of $ad^{\wedge 4}$ brings into the discussion only one further family of representations beyond those that were mastered within the discussion (not reviewed here, but see [8]) of $ad^{\otimes 3}$. We denote this family as $X_4$ (table 2).

The two irreps, (100200) and (020010), that are listed for $\mathfrak{e}_6$ are related by diagram automorphisms as are (0022), (2002) and (2020), for $\mathfrak{d}_4$. The dimension formula

$$d_4(D) = \frac{1}{4!} D(D-1)(D-3)(D-14) \tag{10}$$

already indicates that $\mathfrak{a}_1$ and $\mathfrak{g}_2$ have no member in the $X_4$ family. Indeed, there do not exist any irreps of $\mathfrak{a}_1$ and $\mathfrak{g}_2$ with $c^{(2)} = 4$. For $\mathfrak{a}_2$ we have again the phenomenon that (10) gives a negative result, here $-70$. Indeed, $\mathfrak{a}_2$ has got exactly one pair of conjugate irreps with $c^{(2)} = 4$ which have dimension 35 + 35. For all irreps listed in table 2, (6) is satisfied.

The dimension formulae given here in (4), (9) and (10) are equivalent to the results given in [7, 8], where other parametrizations of family properties are used: see section 1.4. The $c^{(2)}$ eigenvalues can also be found in these sources.

### *1.3. Summary of new results*

We now turn to the problem of extending uniformity properties for $\mathfrak{g} \in \mathcal{F}$ in the case of $ad^{\otimes 5}, \ldots, ad^{\otimes 9}$. A systematic extension would seem to entail massive computational effort, but confirmation that the nice picture known for $j \leqslant 4$ does not stop at $j = 4$ can be provided.

Looking at $ad^{\wedge j}$ for $j = 2, 3, 4$ motivates easy but compelling conjectures. It is natural to expect that there exist, for higher $j$ values, identifiable families $X_j$ of representations of $\mathfrak{g} \in \mathcal{F}$ occurring in the decomposition of $ad^{\wedge j}$, that they satisfy (6) and that nice dimension formulae exist.

The purpose of this paper then is to attain such knowledge by confrontation of the cases of $j = 5, \ldots, 9$. In fact we are able to provide an identification of the members of families $X_5, \ldots, X_9$ of representations of $\mathfrak{g} \in \mathcal{F}$ that satisfy (6) and establish the dimension formulae

$$d_5(D) = \frac{1}{5!} D(D-3)(D-6)(D^2 - 21D + 8) \tag{11}$$

$$d_6(D) = \frac{1}{6!} D(D-1)(D-10)(D^3 - 34D^2 + 181D - 144) \tag{12}$$

$$d_7(D) = \frac{1}{7!} D(D-2)(D-3)(D-8)(D^3 - 50D^2 + 529D - 120) \tag{13}$$

$$d_8(D) = \frac{1}{8!} D(D-1)(D-3)(D-6)(D^4 - 74D^3 + 1571D^2 - 9994D + 4200) \tag{14}$$

$$d_9(D) = \frac{1}{9!} D(D-1)(D-3)(D-4)(D-14)(D-26)(D^3 - 60D^2 + 491D - 120). \tag{15}$$

We display information in tables 3 and 6 that describe in full the assignments of representations for the members of the families $X_5, \ldots, X_9$ for all Lie algebras $\mathfrak{g} \in \mathcal{F}$. There are various features of these results that need, and will receive, consideration.

1. The occurrence of the quadratic, cubic and quartic polynomials in (11)–(15) which do not have rational factors.

**Table 3.** Irreps related to (11).

| | $\mathfrak{a}_1$ | $\mathfrak{a}_2$ | $\mathfrak{g}_2$ | $\mathfrak{d}_4$ | $\mathfrak{f}_4$ | $\mathfrak{e}_6$ | $\mathfrak{e}_7$ | $\mathfrak{e}_8$ |
|---|---|---|---|---|---|---|---|---|
| $d_5$ | 0 | −64 | −924 | $3 \cdot 3696$ +15092 | 629356 +952952 | $2 \cdot 1559376$ +12514788 | 163601438 +109120648 | 6899079264 |
| $X_5$ | – | (33) | (14) | (0104) ⊕(0140) ⊕(4100) ⊕(2022) | (0030) ⊕(0103) | (030000) ⊕(000300) ⊕(110110) | (0000102) ⊕(0002000) | (00100000) |

**Table 4.** Parameters $D$, $m$ and $\alpha$. See (16) and the text for details.

| | $\mathfrak{a}_1$ | $\mathfrak{a}_2$ | $\mathfrak{g}_2$ | $\mathfrak{d}_4$ | $\mathfrak{f}_4$ | $\mathfrak{e}_6$ | $\mathfrak{e}_7$ | $\mathfrak{e}_8$ |
|---|---|---|---|---|---|---|---|---|
| $D = \dim \mathfrak{g}$ | 3 | 8 | 14 | 28 | 52 | 78 | 133 | 248 |
| $\alpha$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{6}$ | $\frac{1}{9}$ | $\frac{1}{12}$ | $\frac{1}{18}$ | $\frac{1}{30}$ |
| $m$ | $-\frac{4}{3}$ | $-1$ | $-\frac{2}{3}$ | 0 | 1 | 2 | 4 | 8 |

2. The status of the table entries for $d_j$ for each $\mathfrak{g}$ when $j$ exceeds the first $j$-value $j_0$ for which $d_j$ is not positive. For $\mathfrak{a}_2, \mathfrak{g}_2, \mathfrak{d}_4, \mathfrak{f}_4$ we have[4] $j_0 = 3, 4, 7, 10$.
3. The appearance of direct sums of several irreducible representations that are not related by diagram automorphisms. This feature is new compared with the results of [8].
4. The occurrence of negative values of the dimension formulae (11)–(15).
5. The fact that the dimension formulae (11)–(15) give integer results for any integral $D$.
6. The question of whether these patterns extend beyond the members of the family $\mathcal{F}$.

The ensuing material is organized as follows. For comparison with the work of others in section 1.4, we mention parametrizations alternative to $D = \dim \mathfrak{g}$. In section 2, we explain our construction of the dimension formulae (11)–(15). Section 3 then poses the obvious question: are the results discussed here for $\mathfrak{g} \in \mathcal{F}$ universal? The results for $j = 2$ are well known to be universal in that they apply, not only to $\mathfrak{g} \in \mathcal{F}$, but also to all simple $\mathfrak{g}$. To what extent, if any, does a similar statement hold for higher $j$? We are unable to give a systematic algebraic analysis of the situation, but can easily gain some insight into it, by giving an empirical analysis of the cases of the simple Lie algebras $\mathfrak{b}_2, \mathfrak{b}_3, \mathfrak{c}_3, \mathfrak{a}_3, \ldots, \mathfrak{a}_5$. Further insight, partially motivated by the appearance of $D - 6$ factors in $d_5(D)$ and $d_8(D)$, comes from the study in section 4 of the cases of $\mathfrak{a}_1 \oplus \mathfrak{a}_1$, and the corresponding threefold and fourfold direct sum. Section 5 contains a conclusion and a list of the most obvious open questions.

### *1.4. Alternative parametrizations*

In general in our work, we prefer to give formulae for the dimensions and the quadratic Casimir eigenvalues of members of a family of irreps of the Lie algebras in $\mathcal{F}$ as a function of the single parameter $D = \dim \mathfrak{g}$, but other parameters are used in the literature. To aid comparison of our discussion with related work in other sources, we have drawn up table 4.

The parameter $\alpha$ is used in [7, 8], while $m$ is used in [10, 12]. The relation between the different parameters can be obtained from

$$D = \frac{2(3m+7)(5m+8)}{m+4} \qquad \frac{1}{\alpha} = 3(m+2) = h^{\vee} \tag{16}$$

[4] These numbers are related to the highest integer $j$ for which $ad^{\wedge j}$ contains a Casimir eigenspace of eigenvalue $j$ [11].

**Table 5.** The highest positive roots $x_j$ of $\mathfrak{f}_4$ in terms of the simple roots $\alpha_j$ and their height.

| Root | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | Height |
|---|---|---|---|---|---|
| $x_{24}$ | 2 | 3 | 4 | 2 | 11 |
| $x_{23}$ | 1 | 3 | 4 | 2 | 10 |
| $x_{22}$ | 1 | 2 | 4 | 2 | 9 |
| $x_{21}$ | 1 | 2 | 3 | 2 | 8 |
| $x_{20}$ | 1 | 2 | 3 | 1 | 7 |
| $x_{19}$ | 1 | 2 | 2 | 2 | 7 |
| $x_{18}$ | 1 | 2 | 2 | 1 | 6 |
| $x_{17}$ | 1 | 1 | 2 | 2 | 6 |
| $x_{16}$ | 0 | 1 | 2 | 2 | 5 |
| $x_{15}$ | 1 | 1 | 2 | 1 | 5 |
| $x_{14}$ | 1 | 2 | 2 | 0 | 5 |
| $\vdots$ | | | | | $\vdots$ |

where $h^\vee$ is the dual Coxeter number (p 37 of [3]). Also $\alpha$ is related to the eigenvalue of the Casimir operator of the members of the $R_3$ family

$$c^{(2)}(R_3) = 2(1+\alpha). \tag{17}$$

For the exceptional algebras in the last line of the Freudenthal magic square, the $m$ values in table 4 have this interpretation: the division algebra used in their Freudenthal construction has dimension $m$.

## 2. The dimension formulae

In the study of $ad^{\wedge j}$ up to $j = 4$ [8] it was sufficient to identify the irreducible component of the highest weight in the antisymmetric power $ad^{\wedge j}$ and then to determine the direct sum $X_j$ of all irreps that can be obtained from the former by the application of diagram automorphisms. The dimension formulae (9) and (10) then agree with the interpolation polynomial for which $d_j(\dim\mathfrak{g}) = \dim X_j$ for all algebras $\mathfrak{g} \in \mathcal{F}$ for which the corresponding $X_j$ satisfies (6). In our notation, it is already a non-trivial fact that a polynomial in $D = \dim\mathfrak{g}$ is sufficient to parametrize the relevant dimensions for all algebras $\mathfrak{g} \in \mathcal{F}$.

At $j = 5$, however, the same strategy does not result in any simple dimension formula at all. The solution is to modify the strategy and choose $X_j$ to be the entire Casimir eigenspace of $ad^{\wedge j}$, i.e. the maximal $X_j$ that satisfies (6) or, if there is no non-trivial subspace with this property when the algebra $\mathfrak{g}$ has dropped out of the full picture, to choose a suitable direct sum of (other) irreps of $\mathfrak{g}$ that satisfy (6). We have arrived at this result purely empirically, searching for simple and in particular polynomial dimension formulae, and we have found the following rule[5] which characterizes the direct summands of $X_j$.

The rule specifies how to select $j$ distinct roots of $\mathfrak{g}$ whose sum is the highest weight of an irrep that is contained in $ad^{\wedge j}$. Whenever it happens that an algebra $\mathfrak{g}$ has not yet dropped out of the full picture (as explained above), the rule finds all irreps that both occur in $ad^{\wedge j}$ and have $c^{(2)} = j$. If the algebra has dropped out, the rule finds some other irreps in $ad^{\wedge j}$.

We explain the procedure for $\mathfrak{f}_4$ whose roots are given in table 5. Consider the root lattice of $\mathfrak{f}_4$, drawn as a directed graph in figure 1. The vertices correspond to the roots $x_k$ and are numbered as in the table. There is a directed arrow from $x_k$ to $x_\ell$, denoted by a pair $(x_k, x_\ell)$, if and only if $x_\ell = x_k + \alpha$ for some simple root $\alpha$.

[5] We thank J Landsberg for bringing the article [13] to our attention in which the Casimir $j$ eigenspace of $ad^{\wedge j}$ is characterized by an algebraic condition equivalent to the rule we state here.

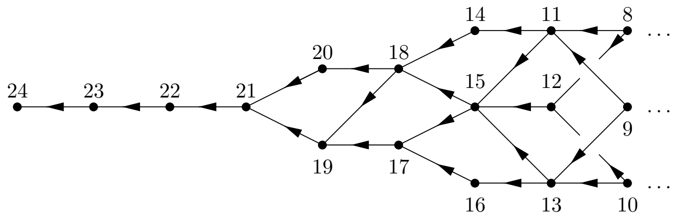


**Figure 1.** A part of the root lattice of $\mathfrak{f}_4$ as a directed graph. The vertices correspond to the roots $x_k$ and are numbered as in table 5.

Let $X = \{x_1, x_2, \ldots\}$ denote the set of all roots. For $j = 1, 2, \ldots$ consider subsets $S \subset X$ of cardinality $|S| = j$. A subset $S$ is called *admissible* if for each $x \in S$ and each arrow $(x, y)$ in the graph, we also have $y \in S$.

Observe that in the $\mathfrak{f}_4$ example, the highest root $x_{24}$ is contained in any non-empty admissible subset. Similarly, the second highest root $x_{23}$ is contained in any admissible set of cardinality at least 2, and so on.

**Observation.** Given an admissible subset $S$, $|S| = j$, the weight

$$w := \sum_{x \in S} x \tag{18}$$

is the highest weight of an irrep of $\mathfrak{g}$ which is contained in $ad^{\wedge j}$. Any irrep of $c^{(2)} = j$ that occurs in $ad^{\wedge j}$ can be found from this rule.

Given the root lattices of the algebras $\mathfrak{g} \in \mathcal{F}$, we can use this rule in order to obtain a list of all irreps that are both contained in $ad^{\wedge j}$ and also have $c^{(2)} = j$. This information forms the basis for the higher dimension formulae.

A point for $\mathfrak{d}_4$ regarding admissible sets is worth noting. For $\mathfrak{d}_4, d_4(28) = 3 \cdot 3675$ involving a triple of irreps with $c^{(2)} = 4$ which occur in $ad^{\wedge 4}$. But $\mathfrak{d}_4$ also has $1925 = (0300)$ with $c^{(2)} = 4$ which is not part of $ad^{\wedge 4}$. If we examine the admissible sets for $\mathfrak{d}_4$ at $j = 4$, we can finds sets for the 3675-dimensional irreps, but not for 1925.

We obtain the dimension formula (11),

$$d_5(D) = \tfrac{1}{5!} D(D-3)(D-6)(D^2 - 21D + 8) \tag{19}$$

as the interpolation polynomial using the data for six algebras of the family $\mathcal{F}$ from table 3.

If we hoped that the right-hand side of (19) is the product of factors linear in $D$, such as the formula for $d_j(D)$ for lower $j$, then we would have been disappointed. However, the expectation was based on viewing these formulae, as in [10], in relation to the Weyl formula for the dimensions of irreps of Lie algebras, and that view is valid as long as the families $X_j$ involve only irreps (up to diagram automorphisms), i.e. for $j \leqslant 4$. But it is not valid for $j = 5$ and the assignments for $X_5$ already made, and so the basis for the hope has gone.

In case it might be thought that the use of the parameter $m$ in table 4 might improve the status of (19), we note that (16) implies

$$D^2 - 21D + 8 = \frac{6(15m^2 + 67m + 68)(10m^2 + 27m + 28)}{(m+4)^2} \tag{20}$$

in which each one of the quadratic expressions in view has discriminant 409 and does not have rational factors.

The dimension formula (19) gives negative values for $\mathfrak{a}_2$ and $\mathfrak{g}_2$ (table 3), which in each case refer to the unique irrep of the Lie algebra in question with $c^{(2)} = 5$.

Although it would not be correct to assign 924 of $\mathfrak{g}_2$ to $X_5$, since it does not occur in the decomposition for $\mathfrak{g}_2$ of $ad^{\wedge 5}$, the situation is similar to that found for $\mathfrak{a}_2$ at the previous stage: we found that there is no proper member of the $X_4$ family for $\mathfrak{a}_2$, but one with the correct value of $c^{(2)}$ and the negative of the correct dimension. Such things are prevalent also in the work of [7] and [8], but not explained.

As one goes to higher $j$ in an effort to carry the search for families and for dimension formulae for them further, one expects more and more algebras to drop out of the full picture much in the way that $\mathfrak{a}_2$ did beyond $j = 2$ and $\mathfrak{g}_2$ did beyond $j = 3$. This will happen just because $\dim ad^{\wedge j}$ eventually becomes too small to contain any irrep of $c^{(2)} = j$. We also expect that for Lie algebras that have dropped out of the full picture, i.e. out of correctly assigning members to families, use of dimension formulae will continue to yield in modulus representations carrying the correct Casimir eigenvalue for the family in question.

As we are looking for a dimension formula $d_j(D)$ which is a polynomial of degree $j$ in $D$, we need $j + 1$ Lie algebras to fix its coefficients and then another Lie algebra in order to confirm that the dimension formula contains non-trivial information.

Table 6 shows the values $d_j(D)$ of the dimension formulae (12)–(15), $j = 6, \ldots, 9$, and the assignment of representations $X_j$. For $\mathfrak{a}_1$, $d_6 = -7$ and the seven-dimensional irrep of $\mathfrak{a}_1$ has $c^{(2)} = 6$, but $d_7 = d_8 = d_9 = 0$ as expected.

For $j = 6, 7$, we have obtained the dimension formulae (11) and (12) as the interpolation polynomial for the eight values $d_j(D)$ where $D = \dim \mathfrak{g}$ for the eight Lie algebras $\mathfrak{g} \in \mathcal{F}$. Whenever $ad^{\wedge j}$ contains irreps of $c^{(2)} = j$, then we choose $X_j$ to be their direct sum and $d_j(\dim \mathfrak{g}) = \dim X_j$. Whenever $ad^{\wedge j}$ does not contain any irreps of $c^{(2)} = j$, so that $\mathfrak{g}$ has dropped out of the full picture, then $d_j(\dim \mathfrak{g})$ is the sum or difference of the dimensions of all irreps of $\mathfrak{g}$ with $c^{(2)} = j$. In this case, the signs of the summands are chosen by trial and error so that we obtain a 'simple' dimension formula, i.e. one which has as many linear factors as possible, which has only 'small' coefficients and for which $d_j(D)$ is an integer for any integral $D$. By experimentation with these interpolations, we always find a unique choice of signs which dramatically, absolutely dramatically, simplifies the interpolation polynomial.

For $j = 7$, it is of course a trivial fact that we can use the data of eight algebras in order to uniquely fix a polynomial $d_j(D)$ of degree 7. The simplicity of the resulting formula (13) is, however, a highly non-trivial property.

Comparing the dimension formulae (10)–(13), the following general pattern emerges: we have $d_j(0) = 0$, and the leading term is $\frac{1}{j!}D^j$. For $j = 8$, we now assume these two conditions and can therefore determine a polynomial of degree 8 from only seven additional data points. We employ all algebras $\mathfrak{g} \in \mathcal{F}$ except for $\mathfrak{d}_4$ and obtain (14).

For $\mathfrak{d}_4$, we discover the following exception from the rules stated so far. The dimension formula (14) yields $d_8(28) = -554400$. There exist indeed representations of $\mathfrak{d}_4$ with $c^{(2)} = 8$, namely $(0106) \oplus (0160) \oplus (6100)$ of dimension $3 \cdot 15015$ and $(1213) \oplus (1231) \oplus (3211)$ of dimension $3 \cdot 169785$, and indeed $d_8(28)$ is the negative sum of their dimensions. However, $\mathfrak{d}_4$ has got further representations with $c^{(2)} = 8$ that do not play any role in the dimension formula $d_8(D)$, namely $(0044) \oplus (4004) \oplus (4040)$ with dimension $3 \cdot 35035$.

For $j = 9$, we again assume the two conditions and determine a polynomial of degree 9 from eight data points, making use of all eight algebras $\mathfrak{g} \in \mathcal{F}$.

## 3. Dimension formulae for simple $\mathfrak{g}$ not in $\mathcal{F}$

The formulae (4), (9), (10), (11)–(15) have been derived and discussed in the context of the extended family $\mathcal{F}$ of simple Lie algebras $\mathfrak{g}$ that include the exceptionals.

**Table 6.** Irreps related to (12)–(15).

| | $\mathfrak{a}_2$ | $\mathfrak{g}_2$ | $\mathfrak{d}_4$ | $\mathfrak{f}_4$ | $\mathfrak{e}_6$ | $\mathfrak{e}_7$ | $\mathfrak{e}_8$ |
|---|---|---|---|---|---|---|---|
| $d_6$ | 28<br>+28 | −1547 | $3 \cdot 1386$ | 7113106<br>+1850212 | $2 \cdot 47783736$<br>+64205141 | 4260501784<br>+1063409347 | 267413986840 |
| $X_6$ | (06)<br>⊕(60) | (23) | (0006)<br>⊕(0060)<br>⊕(6000) | (0112)<br>⊕(1005) | (120100)<br>⊕(010210)<br>⊕(201020) | (0001101)<br>⊕(0000013) | (01000001) |
| $d_7$ | 0 | 1254<br>−748 | $-3 \cdot 23400$<br>−114400 | 1264120<br>+19214624<br>+15997696 | $2 \cdot 466237200$<br>+221077350<br>+152423700 | 2457458575<br>+44406104600<br>+39557939200 | 5006235840320<br>+3754721200320 |
| $X_7$ | – | (07),<br>(40) | (0204)<br>⊕(0240)<br>⊕(4200)<br>⊕(2122) | (0007)<br>⊕(1014)<br>⊕(0202) | (101120)<br>⊕(211010)<br>⊕(020200)<br>⊕(300031) | (0000004)<br>⊕(0001012)<br>⊕(0010200) | (10000002)<br>⊕(02000000) |
| $d_8$ | −125 | 3003 | $-3 \cdot 15015$<br>$-3 \cdot 169785$ | 65609375<br>+15611882<br>+13530946 | 3863940795<br>$+2 \cdot 764156250$<br>$+2 \cdot 1533061530$<br>+133024320 | 135058673750<br>+848520798125<br>+204501797500 | 35361935272950<br>+212182409960235 |
| $X_8$ | (44) | (16) | (0106)<br>⊕(0160)<br>⊕(6100)<br>⊕(1213)<br>⊕(1231)<br>⊕(3211) | (1104)<br>⊕(0016)<br>⊕(0300) | (111110)<br>⊕(002030)<br>⊕(302000)<br>⊕(200131)<br>⊕(310021)<br>⊕(400040) | (0001003)<br>⊕(0010111)<br>⊕(0100300) | (00000003)<br>⊕(11000001) |
| $d_9$ | 80<br>+80 | 0 | $-3 \cdot 215600$<br>+245700 | 64194312 | $2 \cdot 1621233900$<br>$+2 \cdot 5284021600$<br>$+2 \cdot 7587880300$<br>+16162889600 | 3581756027850<br>+3334268437500<br>+6557368727910<br>+539884745400 | 2624940724551600<br>+3500209714601600 |
| $X_9$ | (17)<br>⊕(71) | – | (0322)<br>⊕(2302)<br>⊕(2320),<br>(3033) | (0106) | (300140)<br>⊕(410030)<br>⊕(101041)<br>⊕(401011)<br>⊕(012020)<br>⊕(202100)<br>⊕(210121) | (0010102)<br>⊕(0020020)<br>⊕(0100211)<br>⊕(1000400) | (01000002)<br>⊕(20100000) |

It is known, however, that (4) is universal: a well-defined representation $X_2$ of each simple $\mathfrak{g}$ has its dimension given by (4) and the eigenvalue $c^{(2)}(X_2) = 2$ of its quadratic Casimir operator. It is natural to ask if the other dimension formulae (9), (10), (11)–(15) are likewise universal, and, if not, what, if anything, they can tell us for $\mathfrak{g} \notin \mathcal{F}$.

No systematic algebraic approach is available, but a large body of data can readily be assembled, e.g., using our programs, MAPLE, and references such as [4, 5]. Some indication of the limitations, if any, of the applicability of (9)–(15) to $\mathfrak{g} \notin \mathcal{F}$ can certainly be gained by reference to the cases of $\mathfrak{b}_2$ ($\cong \mathfrak{c}_2$), $\mathfrak{b}_3$, $\mathfrak{c}_3$, $\mathfrak{a}_3, \ldots, \mathfrak{a}_5$.

The entries of table 7 for each $\mathfrak{g}$ and each $X_j$ show representations of $\mathfrak{g}$—often direct sums of irreps—all with the $c^{(2)}(X_j) = j$. The dimension formulae $d_j(D)$, $D = \dim \mathfrak{g}$, yield sums or differences of the dimensions of the irreducible components of $X_j$. Note that the feature that $d_j(D)$ gives a difference of the dimensions of irreps was, for $\mathfrak{g}_2$, first encountered at $j = 7$.

**Table 7.** The dimension formulae $d_j(D)$ for some algebras $\mathfrak{g}$ and the corresponding representations $X_j$ in highest weight notation.

| | $\mathfrak{b}_2$ | $\mathfrak{b}_3$ | $\mathfrak{c}_3$ | $\mathfrak{a}_3$ | $\mathfrak{a}_4$ | $\mathfrak{a}_5$ |
|---|---|---|---|---|---|---|
| $d_2$ | 35 | 189 | 189 | $45+\overline{45}$ | $126+\overline{126}$ | $280+\overline{280}$ |
| $X_2$ | (12) | (102) | (210) | (210) $\oplus$(012) | (2010) $\oplus$ (0102) | (20010) $\oplus$ (01002) |
| $d_3$ | 30 | 294 +616 | 385 +525 | $35+\overline{35}$ +175 | 1024 $+224+\overline{224}$ | 3675 $+840+\overline{840}$ |
| $X_3$ | (30) | (004) $\oplus$(202) | (030) $\oplus$(301) | (400) $\oplus$(004) $\oplus$(121) | (1111) $\oplus$(3010) $\oplus$ (0103) | (11011) $\oplus$(30100) $\oplus$ (00103) |
| $d_4$ | −105 | 1386 +819 | 2205 | 105 | $1701+\overline{1701}$ +1176 $+126+\overline{126}$ | $12250+\overline{12250}$ +6720 $+1050+\overline{1050}$ |
| $X_4$ | (14) | (104) $\oplus$(310) | (121) | (040) | (2201) $\oplus$ (1022) $\oplus$(0220) $\oplus$(5000) $\oplus$ (0005) | (21101) $\oplus$ (10112) $\oplus$(02020) $\oplus$(41000) $\oplus$ (00014) |
| $d_5$ | −84 −154 | 378 | 2457 −2079 | −189 $-\overline{189}$ −729 | 3024 $+\overline{3024}$ | $36750+\overline{36750}$ +34496 $+12936+\overline{12936}$ $+462+\overline{462}$ |
| $X_5$ | (06) $\oplus$(32) | (500) | (022), (501) | (501) $\oplus$(105) $\oplus$(222) | (1310) $\oplus$(0131) | (01121) $\oplus$ (12110) $\oplus$(20202) $\oplus$(32001) $\oplus$ (10023) $\oplus$(60000) $\oplus$ (00006) |
| $d_6$ | – | −9009 −4312 | −11319 −3003 +1001 | −735 −875 $-\overline{875}$ | $-8624-\overline{8624}$ $-924-\overline{924}$ $+1176+\overline{1176}$ | 169785 $+43120+\overline{43120}$ $+25200+\overline{25200}$ |
| $X_6$ | – | (114) $\oplus$(320) | (321), (610), (004) | (141) $\oplus$(412) $\oplus$(214) | (3202) $\oplus$ (2023) $\oplus$(6001) $\oplus$ (1006) $\oplus$(0500) $\oplus$ (0050) | (11211) $\oplus$(23010) $\oplus$ (01032) $\oplus$(03200) $\oplus$ (00230) |

If we extend table 7 to $\mathfrak{b}_4$ and $\mathfrak{c}_4$, we encounter at $j = 5$ the same exception from the rules that we have already seen for $\mathfrak{d}_4$ at $j = 8$, namely that the algebra has already dropped out of the full picture (as explained above), and there exist many irreps with $c^{(2)} = j$ only some of which are relevant for the dimension formula.

It is a striking observation that all the simple Lie algebras of table 7 fit into the general pattern. In particular, the fact that $\mathfrak{b}_2$ does not have any irrep of $c^{(2)} = 6$ can be seen as an 'explanation' of the linear factor $D - 10$ in (12). It is then a natural question to ask which are the Lie algebras of dimensions 1, 2, 4, 6 and 26 that cause the other integer roots of the dimension formulae (11)–(15).

## 4. Some further studies

### 4.1. The factor $(D - 6)$ in $d_5(D)$ and $d_8(D)$

To account for the presence of the factors $(D-6)$, consider the case of $\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_1$, employing the Cartan matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{21}$$

**Table 8.** Data for $\mathfrak{a}_1 \oplus \mathfrak{a}_1$.

| $c^{(2)} = n$ | Irreps | $d_n(6)$ |
|---|---|---|
| 1 | $(2,0) \oplus (0,2)$ | 6 |
| 2 | $(2,2)$ | 9 |
| 3 | $(4,0) \oplus (0,4)$ | $-10$ |
| 4 | $(4,2) \oplus (2,4)$ | $-30$ |
| 5 | None | 0 |
| 6 | $(6,0) \oplus (0,6), (4,4)$ | 11 |
| 7 | $(6,2) \oplus (2,6)$ | $-42$ |
| 8 | None | 0 |
| 9 | $(6,4) \oplus (4,6)$ | 70 |

**Table 9.** Data for $\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1$.

| $c^{(2)} = n$ | Irreps | $d_n(9)$ |
|---|---|---|
| 3 | $(2,2,2), 3 \cdot (4,0,0)$ | $12 = 27 - 3 \cdot 5$ |
| 4 | $6 \cdot (4,2,0)$ | $-90 = -6 \cdot 15$ |
| 5 | $3 \cdot (4,2,2)$ | $-135 = -3 \cdot 45$ |
| 6 | $3 \cdot (4,4,0), 3 \cdot (6,0,0)$ | $54 = 3 \cdot 25 - 3 \cdot 7$ |
| 7 | $6 \cdot (6,2,0), 3 \cdot (4,4,2)$ | $3 \cdot 75 - 6 \cdot 21$ |
| 8 | $3 \cdot (6,2,2)$ | $-189 = -3 \cdot 63$ |
| 9 | $6 \cdot (6,4,0), (4,4,4)$ | $85 = 6 \cdot 35 - 125$ |

and a Cartan–Killing form with no relative scaling of the two $\mathfrak{a}_1$ summands, so that the algebra has an $S_2$ group of diagram automorphisms.

Let $(j, k)$ denote the irrep of dimension $(j+1)(k+1)$, so that $ad = (2,0) \oplus (0,2)$. We list in table 8 irreps of $\mathfrak{a}_1 \oplus \mathfrak{a}_1$ with integer eigenvalues $n$ of the quadratic Casimir operator, with $d_n(6)$ alongside for comparisons of the type systematically made in previous cases.

The entries for $c^{(2)} = 5$ and $c^{(2)} = 8$ explain the $(D-6)$ factors in $d_5(D)$ and $d_8(D)$, and all the other entries follow precisely a now familiar pattern. Only one entry needs any comment:

$$d_6(6) = 11 = 25 - 2 \cdot 7 \tag{22}$$

where $25 = \dim(4,4)$, $7 = \dim(6,0) = \dim(0,6)$.

We also note that all the irreps that feature here are either self-conjugate or else occur as conjugate pairs, as the $S_2$ invariance of $ad$ requires.

### *4.2.* $\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1$

In this case we use as Cartan matrix twice the unit matrix, again with no relative scales, so that the algebra has a group $S_3$ of diagram automorphisms. Table 9 displays data about all the irreps with integral values of the Casimir operator. The notation $(j, k, l)$ denotes the irrep with dimension $(j+1)(k+1)(l+1)$, so that $ad = (2,0,0) \oplus (0,2,0) \oplus (0,0,2)$. To keep the displays as brief as is reasonable, the notation $r \cdot (a, b, c)$ denotes the direct sum of all $r$ distinct permutations of $(a, b, c)$. In view of the automorphism group $S_3$, we may have $r = 3$ and $r = 6$. Again we can check that all the data conform to the expected pattern.

*4.3.* $\mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1$

This example was treated to see one further automorphism group at work. But few surprises were expected or found. Everything is in full accord with expectation. We do not display the data that would make a table like tables 8 and 9, but note only the situation for $d_9(12)$. The set of irreps, in notation similar to that used in previous subsections, that have $c^{(2)} = 9$ is

$$12 \cdot (6, 4, 0, 0), \quad 4 \cdot (4, 4, 4, 0), \quad 4 \cdot (7, 1, 1, 1), \quad 4 \cdot (6, 2, 2, 2) \tag{23}$$

with dimensions $12 \cdot 35, 4 \cdot 125, 4 \cdot 64, 4 \cdot 189$, and we have

$$d_9(12) = -836 = 12 \cdot 35 - 4 \cdot (189 + 125). \tag{24}$$

We note that the resolution (24) does not employ the irreps $4 \cdot (7, 1, 1, 1)$, but such an omission is also familiar in previous cases. There is some curious numerology in the $n = 9$ case: $64 = (189 - 125)$, and similar things are seen for lower $n$ cases.

But we leave the analysis here, without including much additional data with features that are of a qualitatively similar nature to what has been presented. The fact, however, is that everything follows a coherent if far from understood pattern.

## 5. Conclusion

We have extended the dimension formulae of [8] for a particular family up to the ninth power of the adjoint representation. The formulae (11)–(15) describe a further striking uniformity of the Lie algebras of the exceptional series and our results of sections 3 and 4 indicate even a uniformity beyond that. The formulae were obtained by inspection of a large amount of data from tables and from computer calculations and finally by a considerable amount of trial and error until we had found the appropriate rules that give rise to 'simple' formulae. The fact that 'easy' formulae such as (11)–(15) with coefficients smaller than a few thousand give rise to integers up to 16 digits which precisely correspond to dimensions of representations of the exceptional Lie algebras, deserves to be seen as of real significance.

We conclude by listing some particular observations and first ideas that come to mind.

- The formulae (11)–(15) for $d_j(D)$ are polynomials in $D = \dim \mathfrak{g}$. From [8], we might have expected only rational functions in $\alpha$, $m$ or $D$. In particular, $\dim Y_j$ in the notation of [8] ($Y_j$ is the highest weight component of the $j$th totally symmetric tensor power of $ad$) is not a polynomial in $D$.
- With the rational functions of [8], one can search for those values of the parameter $\alpha$ for which the result is an integer and thus obtain a list of all algebras that conform to the family pattern. In our case, however, the formulae for $d_j(D)$ give integer results for any integral $D$.
- Whenever the dimension formula ceases to give a strictly positive answer and $ad^{\wedge j}$ does not contain any representation of the desired $c^{(2)}$, we can successfully describe the phenomenology of the situation, but do not have a satisfactory explanation of why it occurs.
- The leading coefficient of the formula for $d_j(D)$ is $1/j!$. This looks like a growth rate of the dimension $d_j(D)$ of the family member $X_j$ if the dimension $D$ of the underlying Lie algebra $\mathfrak{g}$ tends to infinity.
- Our data confirm that the dimension formulae extend to other simple Lie algebras not in the family $\mathcal{F}$ and beyond that also to some non-simple Lie algebras. A particularly strong indication for this is the fact that the algebras $\mathfrak{b}_2$ and $\mathfrak{a}_1 \oplus \mathfrak{a}_1$ 'explain' some of the integer roots of the polynomials (11), (12) and (14).

It is an interesting question as to whether one can identify for each linear factor $(D - m)$ of the dimension formulae $d_j(D)$ a Lie algebra of dimension $m$ for which there exists no irrep with $c^{(2)} = j$. For large $j$, however, there is hardly any simple Lie algebra other than $\mathfrak{a}_1$. It is therefore crucial to go beyond simple Lie algebras and to include more examples in order to prove or disprove the conjecture. In this context, it is a striking observation that $d_9(D)$ of (15) has so many linear factors.

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## References

[1] Freudenthal H 1964 Lie groups in the foundations of geometry *Adv. Math.* **1** 145–96

[2] Faulkner J R and Ferrar J C 1977 Exceptional Lie algebras and related algebraic and geometric structures *Bull. Lond. Math. Soc.* **9** 1–35

[3] Frappat L, Sciarrino A and Sorba P 2001 *A Dictionary on Lie Algebras and Superalgebras* (London: Academic)

[4] Slansky R 1981 Group theory for unified model building *Phys. Rep.* **79** 1–128

[5] Cornwell J F 1984 *Group Theory in Physics* vol 2 (London: Academic)

[6] Meyberg K 1984 Spurformeln in einfachen Lie-algebren *Abh. Math. Sem. Hamburg* **54** 177–89

[7] Deligne P 1996 La série exceptionnelle de groupes de Lie *C. R. Acad. Sci., Paris* I **322** 321–6

[8] Cohen A M and de Man R 1996 Computational evidence for Deligne's conjecture regarding the exceptional groups *C. R. Acad. Sci., Paris* I **322** 427–32

[9] van Leeuwen M A and Cohen A M 1992 *LiE, a Package for Lie Group Computation* (Amsterdam: CAN)

[10] Landsberg J M and Manivel L 2001 Triality, exceptional Lie algebras and the Deligne dimension formulas *Preprint* math.AG/0107032
Landsberg J M and Manivel L 2002 Series of Lie groups *Preprint* math.AG/0203241

[11] Malc'ev A I 1945 Commutative subalgebras of semi-simple Lie algebras *Izv. Akad. Nauk. SSR Ser. Math.* **9** 291–300

[12] Westbury B W 2003 R-matrices and the magic square *J. Phys. A: Math. Gen.* at press

[13] Kostant B 1988 The set of Abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations *Int. Math. Res. Notices* **98** 225–52